A NOTE ON THE RECOGNITION OF CODISMANTLABLE GRAPHS AND POSETS

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ABSTRACT. We show that when a graph G is codismantlable, then the codismantling order does not matter from which we conclude an $O(|V|^3/\log|V|)$ running time algorithm for checking codismantlability by using Spinrad's data structure. In addition, we introduce codismantlable posets, and prove that codsimantlability of posets is equivalent to that of their comparability graphs.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean an undirected finite graph without loops or multiple edges. When x and y are two distinct vertices of a graph G, if $N_G[x] \subseteq N_G[y]$, then the vertex x is said to be a *dominated vertex*, whereas the vertex y is called a *codominated vertex* of G, where $N_G(z) := \{u \in V : uz \in E\}$ and $N_G[z] = N_G(z) \cup \{z\}$ are the open and closed neighbourhood sets of the vertex z in G.

There are two graph classes, dismantlable and codismantlable graphs, both of which can be defined by the existence of particular (full/partial) vertex elimination schemes built on removing recursively dominated and codominated vertices respectively.

In detail, a graph G is called a *dismantlable* (or *cop-win*) graph [4] (see also [2] for a recent detailed book on cop-win graphs), if either G has a single vertex or else there exists a dominated vertex x in G for which $G - x$ is dismantlable. It means that when G is dismantlable, there exists an ordering of all of its vertices $v_1, v_2, \ldots, v_n$ such that v_{i+1} is a dominated vertex in $G_i := G - \{v_1, v_2, \ldots, v_i\}$ for each $0 \leqslant i < n$, where $G_0 := G$. Such an ordering of vertices of G is said to be a *dismantling ordering*.

In order to characterize a subclass of vertex decomposable graphs, we introduce the notion of codismantlable graphs in [1]. We call a graph G *codismantlable* if either it is an edgeless graph or else it has a codominated vertex y such that $G - y$ is codismantlable. It means that there is an ordered list $\{u_1, \ldots, u_k\} \subseteq V$ of vertices such that u_{j+1} is a codominated vertex in $G_j := G - \{u_1, \ldots, u_j\}$ for each $0 \leqslant j < k$, where $G_0 := G$, and the graph $G_{k+1} := G - \{u_1, \ldots, u_k\}$ is an edgeless graph. Such a set $\{u_1, \ldots, u_k\}$ is called a *codismantling order* (or shortly a cd-order) for G.

Remark 1.1. We note that there is no direct relation between the classes of codismantlable and dismantlable graphs, so our terminology is just a coincidence stemming from the relevance of their defining relations. Moreover, they are not complementary graph classes, that is, the complement of a codismantlable graph need not be a dismantlable graph and vice versa.

Date: August 9, 2016.

Example 1.2. The wheel graph W_n for $n \geqslant 4$ is an example of a dismantlable graph that is not codismantlable, and the pan graph Pan_m for $m \geqslant 4$ is a codismantlable graph which is not dismantlable (see Figure 1 for $n = m = 5$). Moreover, while W_n is a dismantlable graph, its complement $\overline{W_n}$ is not codismantlable for any $n \geqslant 5$, and similarly, the complement $\overline{\mathrm{Pan}_m}$ is not dismantlable for any $m \geqslant 5$.

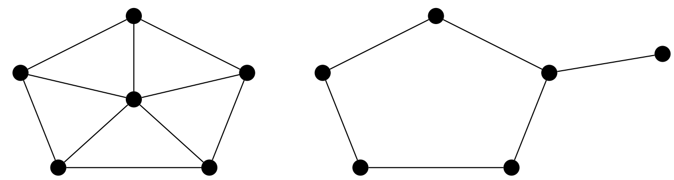

FIGURE 1. The wheel W_5 and the pan Pan_5 graphs.

The notion of *dismantlability* for partially ordered sets (posets for short) firstly introduced by Rival [5] in which he proves that any dismantlable poset has the fixed point property. In analogy with dismantlable posets, we introduce codismantlable posets, and show that codismantlability of posets is equivalent to that their comparability graphs.

2. Recognation of codismantlable graphs

Definition 2.1. We call a pair $\{x, y\}$ of distinct vertices of a graph G a *true-twin*, if $N_G[x] = N_G[y]$ and there exists no vertex $z \in V \backslash \{x, y\}$ satisfying $N_G[z] \subsetneqq N_G[x] = N_G[y]$.

Lemma 2.2. *If x and y are distinct codominated vertices of G such that y is not codominated in $G - x$, then $\{x, y\}$ is a true-twin of G.*

Proof. Since y is codominated in G while it is not codominated in $G - x$, we must have $N_G[x] \subseteq N_G[y]$. On the other hand, since x is codominated in G, there is a vertex $z \in V \backslash \{x\}$ such that $N_G[z] \subseteq N_G[x]$. However, it then follows that $z = y$, since otherwise the vertex y would be codominated in $G - x$. □

Lemma 2.3. *If $\{x, y\}$ is a true-twin of G, then G is codismantlable if and only if $G - x$ is codismantlable.*

Proof. Since the sufficiency is clear, we only prove the necessary part. So, suppose that G is codismantlable, while the graph $G - x$ is not. Observe that any cd-order must contain at least one of the vertices x and y, since $xy \in E$. Furthermore, since $\{x, y\}$ is a true-twin of G, we may assume that any such set contains the vertex x. Among any such cd-orders, we choose one, say $C := \{v_1, v_2, \ldots, v_k\}$, in which the appearance of the vertex x has the lowest possible index, and let $x = v_i$. Note that $i > 1$, since $G - x$ is not codismantlable. If we define $H := G - \{v_1, \ldots, v_{i-2}\}$, observe that x is codominated in H, since $N_H[x] = N_H[y]$, while the vertex v_{i-1} is not codominated in $H - x$. Indeed, if v_{i-1} was codominated in $H - x$, then $C' := \{v_1, v_2, \ldots, v_{i-2}, x, v_{i-1}, v_{i+1}, \ldots v_k\}$ would be a cd-order for G in which the appearance of x comes before than that in C, a contradiction. However, it then follows that $\{x, v_{i-1}\}$ is a true-twin of H by Lemma 2.2, which in turn forces that $\{y, v_{i-1}\}$ is a true-twin of $H - x$, contradicting to the fact that v_{i-1} is not codominated in $H - x$. □

Theorem 2.4. *Let x be a codominated vertex of G. Then G is codismantlable if and only if $G - x$ is codismantlable.*

Proof. Once again we only need to verify the necessary condition of the claim. So, suppose that G is codismantlable, while the graph $G-x$ is not. Moreover, by Lemma 2.3, we may further assume that there exists no vertex $y \in N_G(x)$ such that $\{x, y\}$ is a true-twin of G.

Case 1. There exists at least one cd-order of G containing the vertex x. In this case, as above, we choose a *cd*-order, $C := \{v_1, v_2, \ldots, v_k\}$ of G in which the appearance of the vertex x has the lowest possible index among any such order. We let $x = v_i$ and note that $i > 1$. Since x is codominated in G, so is in $H := G - \{v_1, \ldots, v_{i-2}\}$. On the other hand, the vertex v_{i-1} is not codominated in $H - x$ because of the chosen cd-order C. By Lemma 2.2, the pair $\{x, v_{i-1}\}$ is a true-twin of H, a contradiction.

Case 2. There exists no cd-order of G containing x. Now, let $D := \{u_1, \ldots, u_l\}$ be any cd-order for G. Since $G-D$ is an edgeless graph, we must have $N_G(x) \subseteq D$. On the other hand, since x is codominated in G, there exists $w \in N_G(x)$ such that $N_G[w] \subseteq N_G[x]$. Suppose that $u_j \in N_G(x) \cap D$ is the vertex such that $N_G[u_j] \subseteq N_G[x]$ and j is the greatest index with this property. We define $T := G - \{u_1, \ldots, u_{j-1}\}$, and note that there exists a vertex $z \in T$ satisfying $N_T[z] \subseteq N_T[u_j]$, since u_j is codominated in T. However, this forces $z = x$ by the choice of the vertex u_j, that is, $\{x, u_j\}$ is a true-twin of T. We may then interchange the vertices x and u_j in D by Lemma 2.3 to create a new cd-order for G containing x, a contradiction. □

The counterpart of Theorem 2.4 for dismantlable graphs directly follows from the facts that the induced subgraph $G - x$ is a retract of G whenever x is a dismantlable vertex, and if G is dismantlable, then so is any retract of it (see [2] for details).

Theorem 2.4 naturally produces a greedy algorithm for recognizing codismantlability. For this, we need to check whether G has a codominated vertex or not that consumes $2\sum_{xy\in E} \deg_G(x)\deg_G(y)$ operations. Since we repeat this at most $|V|$ times, the total running time is in $O(|V||E|\Delta(G)^2)$, where $\Delta(G)$ is the maximum degree of G.

Corollary 2.5. *We can recognize for a given graph G of maximum degree Δ, whether it is codismantlable or not in $O(|V||E|\Delta^2)$ time by using the greedy algorithm.*

We recall that Spinrad [6] has introduced a new strategy on the recognition of quasi-triangulated graphs, which is also applicable to dismantlable graphs. His main algorithm is based on constructing lists of pairs with small deficit sets, where $\text{deficit}(u, v) := |N_G(u)\backslash N_G(v)|$ for any pair $\{u, v\}$ of vertices of G. In his language, the existence of a dominated vertex in each phase corresponds to having some edge with deficit at least one from which he obtains an algorithm to recognize dismantlable graphs in $O(|V|^3/\log|V|)$. Together with Theorem 2.4, his approach yields an algorithm for codismantlable graphs having the same running time.

Corollary 2.6. *We can recognize for a given graph G, whether it is codismantlable or not in $O(|V|^3/\log|V|)$ time by using the Spinrad's data structure.*

3. Codismantlable posets

In this section, we show that the codismantlability of posets can be characterized form that of their comparability graphs.

Let $(P, \leqslant)$ be a (finite) partially ordered set (poset for short). For elements $x, y \in P$, we say that y *covers* x in P, denoted by $x \prec y$, if $x < y$ and there exists no elements $z \in P \backslash \{x, y\}$ satisfying $x < z < y$. In such a case, x is called a *lower cover* of y, and y is called an *upper cover* of x in P. An element $x \in P$ is said to be *irreducible* if it has a unique upper or lower cover in P. Dually, we call an element $y \in P$ as *coirreducible* if it is the unique upper or lower cover of an irreducible element in P.

We recall that a poset P is said to be *dismantlable* if its elements can be ordered $x_1, x_2, \ldots, x_n$, where $|P| = n$, such that x_j is an irreducible elements of the poset $P - \{x_i \colon i < j\}$ for each $1 \leqslant j \leqslant n-1$.

In an analogy with dismantlable posets, we next introduce the class of codismantlable posets.

Definition 3.1. We call a poset P *codismantlable*, if there exists a sequence $C := \{c_1, c_2, \ldots, c_k\}$ of elements of P such that c_i is coirreducible in $P_{i-1} := P - \{c_1, c_2, \ldots, c_{i-1}\}$ for any $1 \leqslant i \leqslant k$, where $P_0 := P$, and $P_k := P - C$ is an anti-chain.

We recall that for a poset $(P, \leqslant)$, the *comparability graph* $\mathrm{Comp}(P)$ of P is the (undirected) graph whose vertices are the elements of P and edges are those pairs $\{x, y\}$ of elements such that $x < y$ in P.

Theorem 3.2. *Let P be a poset. Then P is a codismantlable poset if and only if* $\mathrm{Comp}(P)$ *is a codismantlable graph.*

Proof. In view of Theorem 2.4, the claim follows at once if we verify that an element x of P is coirreducible if and only if it is a codominated vertex of $\mathrm{Comp}(P)$.

Assume first that x is coirreducible in P. So, there is an irreducible element, say $z \in P$, such that x is the unique upper or lower cover of z in P. In either case, observe that every element of P which is comparable to z must be comparable to x. It means that $N_{\mathrm{Comp}(P)}[z] \subseteq N_{\mathrm{Comp}(P)}[x]$; hence, x is codominated in $\mathrm{Comp}(P)$.

Conversely, suppose that x is a codominated vertex of $\mathrm{Comp}(P)$. Thus there exists a vertex $y \in P \backslash \{x\}$ satisfying $N_{\mathrm{Comp}(P)}[y] \subseteq N_{\mathrm{Comp}(P)}[x]$. Since x and y are comparable and distinct, we may assume that $y < x$ in P (the case $x < y$ can be treated similarly). Note that the element y need not be irreducible in P. On the contrary, we claim that x is coirreducible. Consider a maximal (saturated) chain $M : y < z_1 < z_2 < \ldots < z_k < x$ within the interval $[y, x]$ for some $k \geqslant 0$. Since M is maximal, x is an upper cover of z_k in P. If u is another upper cover of z_k, we then have $y \leqslant u$, that is, $u \in N_{\mathrm{Comp}(P)}[y] \subseteq N_{\mathrm{Comp}(P)}[x]$. However, if $x < u$, then u can not cover z_k in P, and if $u < x$, then M can not be maximal. Therefore we must have $u = x$. It follows that x is the unique upper cover of z_k implying that x is coirreducible in P as claimed. $\square$

We remark that there is a similar characterization of dismantlable posets. Ginsburg [3] (compare to Corollary 2.5 of [3]) shows that a poset P is dismantlable if and only if its comparability graph $\mathrm{Comp}(P)$ is dismantlable, while the strategy of our proof does not work in the dismantlable case, since a dominated vertex of $\mathrm{Comp}(P)$ need not be necessarily an irreducible element of P.

Note also that one of the consequence of Theorem 3.2 is that in the language of posets, the dismantlability and codismantlability are not related. For instance, the $2n$-wheel W_{2n} and the $2n$-pan Pan_{2n} for any $n \geqslant 3$ are comparability graphs such that the corresponding poset of W_{2n} is dismantlable, while it is not codismantlable, and the poset of Pan_{2n} is codismantlable, while it is not dismantlable.

While Theorem 3.2 provides a characterization of codismantlability of posets in terms of those of comparability graphs, in the case of dismantlability, Ginsburg [3] finds a way to characterize the dismantlability of graphs in terms of those of associated posets, namely the face posets of clique complexes $\mathrm{Cl}(G)$ of graphs. We recall that for any simplicial complex Δ, its *face poset* $P(\Delta)$ is the poset on the faces of Δ ordered with respect to the inclusion. Ginsburg [3] shows that dismantlablity of a graph G is equivalent to that of $\overline{P(\mathrm{Cl}(G))}$, where $\overline{P(\mathrm{Cl}(G))}$ is obtained from $P(\mathrm{Cl}(G))$ by removing the empty clique of G (see Theorem 2.4 of [3]). However, such a characterization does not valid for codismantlability of G in terms of $P(\mathrm{Cl}(G))$ or even $P(\mathrm{Ind}(G))$, where $\mathrm{Ind}(G)$ is the independence complex of G. For instance, the graph Pan_4 is codismantlable, while none of $P(\mathrm{Cl}(\mathrm{Pan}_4))$ and $P(\mathrm{Ind}(\mathrm{Pan}_4))$ is. In this guise, we leave it open the question of finding (if possible) a poset $P(G)$ for any graph G such that codismantlabilities of G and $P(G)$ are equivalent.

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